Flexible Transitional Dynamics in a Non-Scale Fully Endogenous Growth Model

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April 21, 2015

This paper develops a non-scale growth model of physical capital accumulation and two types of lab-equipment R&D, where both the intensive and extensive margins of growth are fully endogenous. We study analytically the long-run equilibrium stability and transitional dynamics properties of the model, and establish meaningful sufficient conditions for saddle-path stability. We relate the different combinations of initial conditions of the dynamical system with the observation of monotonic versus non-monotonic transitional dynamics. Our model is able to predict monotonic, hump-shaped and inverted hump-shaped trajectories, therefore encompassing the evidence reported by the empirical literature for distinct subsets of countries.

– Preliminary draft version –

Keywords: endogenous growth, non-monotonic transitional dynamics, R&D, lab equipment, physical capital.

JEL Classification: O41, O31

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1. Introduction

The purpose of this paper is twofold. On one hand, we study analytically the long-run equilibrium stability and transitional dynamics properties of the R&D-based growth model of the lab-equipment type. This is an apparently overlooked topic in the endogenous growth literature in favor of the models of the concurrent type, featuring a knowledge-driven R&D specification. On the other hand, we carry out this study in the context of the flexible transitional dynamics that arise in the presence of a multi-dimensional stable manifold, motivated by the empirical evidence that points out the existence of non-monotonic transitional dynamics phenomena. Notable empirical patterns are: (i) the hump-shaped or inverted hump-shaped behaviour of the transitional economic growth rates (see, e.g., Papageorgiou and Perez-Sebastian, 2006; Fiaschi and Lavezzi, 2003a, 2007; see also Figure 1) and saving rates (Maddison, 1992; Loyaza, Schmidt-Hebbel, and Serven, 2000); (ii) the non-linear relationship between the stock of physical capital and the number of firms over transition (Gil, 2010); and (iii) the time-variable and sector-specific speed of convergence (Bernard and Jones, 1996). Bearing the above in mind, this paper develops a non-scale fully endogenous growth model of physical capital accumulation and two types of R&D – vertical (increase of product quality) and horizontal (creation of new products) –, both under a lab-equipment specification.

[Figure 1 goes about here]

The concern with the stability properties of the long-run equilibrium in R&D growth models and the characteristics of their transitional dynamics has occupied a significant strand of the endogenous growth literature. Several papers devoted efforts to study in that regard the seminal models by Romer (1990), Aghion and Howitt (1992) and Jones (1995), and a variety of extensions of those models (e.g., Arnold, 1998, 2000, 2006; Eicher and Turnovsky, 2001; Kosempel, 2004; Gómez, 2005; Arnold and Kornprobst, 2008; Sequeira, 2011; Growiec and Schumacher, 2013; Sequeira, Lopes, and Gomes, 2014). All these models are of the knowledge-driven type. In contrast, our paper focuses on the equilibrium and transitional dynamics properties of an R&D growth model of the lab-equipment type, as laid out originally by Rivera-Batiz and Romer (1991) and Barro and Sala-i-Martin (2004) (with first edition in 1995). Indeed, from the perspective of the technology of R&D, two polar cases can be considered (Rivera-Batiz and Romer, 1991): the knowledge-driven case, which assumes that human capital and knowledge are the only inputs to R&D activities; and the lab-equipment case, which assumes that the technology for R&D is the same as the technology for final-good production, and thus human capital, raw labour, and capital goods are all productive in R&D activities. In this latter setting, physical capital accumulation and R&D complement each other, meaning that capital accumulation relates to R&D more closely than in the knowledge-driven setting. This allows one to address the empirical evidence that shows an important interconnection between physical and technological inputs along growth processes (e.g., Bernard and Jones, 1996; Papageorgiou and Perez-Sebastian, 2006).

The consideration of the two types of R&D follows from both a substantive (economic) and formal (technical) argument. As for the former, such a setup enables us to address

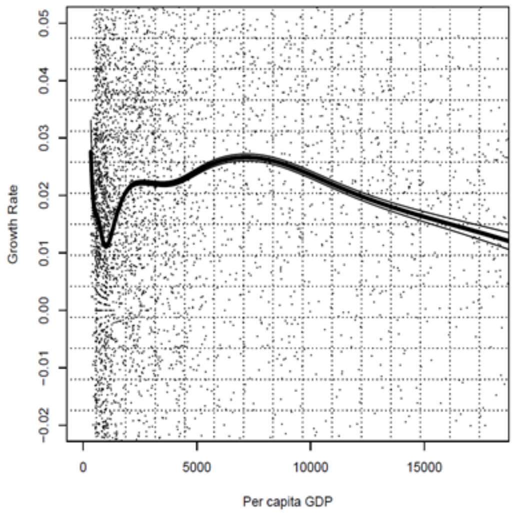

Figure 1: Annual per capita GDP growth rate versus the corresponding annual per capita GDP level, for 122 countries from 1950 to 1998, where the black solid line is a nonparametric regression line. This is a reproduction of the figure in Fiaschi and Lavezzi (2003b, p. 6). These authors use this cross-section of countries over time to estimate transition probabilities in a state space defined in terms of both income levels and growth rates. By keeping track of the growth path of each individual country, they show that different subsets of countries with initial below-world average income – 'low' versus 'middle-low' income countries – may follow, respectively, inverted hump-shaped and hump-shaped transitions.

the general view that industrial growth proceeds both along an intensive (vertical innovation) and an extensive (horizontal innovation) margin in the long run, as well as the evidence relating the initial intensity of use of technological knowledge (i.e., the initial proportion of the intensive vis-à-vis the extensive component of a given technological-knowledge stock) and the transitional growth rates (e.g., Jones and Romer, 2010). In particular, the integration of the two types of R&D in a lab-equipment setup allows us to build a non-scale fully endogenous growth model such that a positive growth rate arises along both the intensive and the extensive margin without relying on a positive (exogenous) population growth rate, i.e., we endogenise *both* margins of growth (and thereby production, the number of firms, and firm size). This stands in contrast with the usual approaches in the knowledge-driven literature, which imply a strictly positive relationship between economic growth (or its extensive margin) and the population growth rate (e.g., Jones, 1995; Dinopoulos and Thompson, 1998; Peretto, 1998; Eicher and Turnovsky, 2001),[1] a result that has not received empirical support (e.g., Dinopoulos and Thompson, 2000; Strulik, Prettner, and Prskawetz, 2013). On the other hand, under this setup, aggregate dynamics is characterised by a third-order dynamical system in appropriately scaled variables, with one jump-like and two state-like variables, where the latter result from the interaction between physical capital and the technological-knowledge stock obtained from the two types of R&D activities. We consider an asymmetry between horizontal and vertical R&D costs (e.g., Howitt, 1999), reflecting the inherently distinct nature of horizontal and vertical innovation, or between technology embodied in new products (more physical) and in improved processes/products (more immaterial). It is this asymmetry that makes the dynamical system of third order, while preserving the fully endogenous-growth result.

As stated earlier, we are interested in the long-run equilibrium stability and transitional dynamics properties of this model. However, an important modelling distinction between the knowledge-driven and the lab-equipment settings is that, in the former, the dynamic general equilibrium is established upon the satisfaction of (at least) two aggregate resource constraints, pertaining to the product market and to the labour (or human capital) market, respectively; in the latter, only the aggregate resource constraint pertaining to the product market is relevant for the derivation of the dynamic general equilibrium. The consequence is that, in the lab-equipment setting, the dynamical-system equations are tied up together by this single aggregate resource constraint, implying that the respective Jacobian matrix is dense, with no or very few null elements, and thus the dynamical system cannot be decoupled; this makes a complete analytical study usually

[1] Jones' (1995) solution to the scale-effect result found in the first-generation endogenous growth models (e.g., Romer, 1990; Aghion and Howitt, 1992) implied that positive economic growth relied on a positive population growth rate (the semi-endogenous growth result). As a reaction, a new generation of endogenous-growth models introduced simultaneous vertical and horizontal R&D as a modelling strategy to remove scale effects while preserving the first-generation result that long-run economic growth has policy-sensitive economic determinants (the fully-endogenous growth result) (e.g., Dinopoulos and Thompson, 1998; Peretto, 1998; Howitt, 1999). However, in these models, the extensive margin of growth still relies on a positive population growth rate. In contrast, following, e.g., Barro and Sala-i-Martin (2004), our approach allows us to remove scale effects and still get the fully endogenous-growth result along both the intensive and the extensive margin.

intractable in the case of high-order dynamical systems (third or above).

Nevertheless, we are able to establish meaningful sufficient conditions, generalisable by a continuity argument, under which the Jacobian of our linearised dynamical system is characterised by two eigenvalues with negative real parts, and thus a saddle-path stable system exists with the stable manifold arising as a two-dimensional locus. That is, we show that, under these conditions, no local indeterminacy can occur in our model, and instability and limit cycles are likewise ruled out; in short, the long-run equilibrium is "well-behaved". But we also show that rich transitional dynamics may arise. Since the dimension of the stable manifold is larger than unity, there are multiple independent sources of stability in the dynamical system, but which interact among themselves. Thus, non-monotonic (but not necessarily oscillatory) trajectories as well as time-variable/sector-specific speeds of convergence can emerge in the state-like variables, thereby introducing important flexibility to the transitional dynamics, as seems to be the case empirically. In particular, by considering a projection of the stable manifold onto the plane of the state-like variables, we relate the different combinations of the initial conditions of the dynamical system (i.e., the initial physical capital-output ratio and the initial intensity of use of technological knowledge) across the phase diagram with the observation of monotonic versus non-monotonic transitional dynamics. Importantly, as regards the latter, our model's predictions encompasse the evidence reported by recent empirical literature for distinct subsets of countries: the inverted hump-shaped (respectively, hump-shaped) trajectory estimated for the subset of initially 'low' ('middle-low') income countries in Fiaschi and Lavezzi's (2003, 2007) sample corresponds in our model to the transition of the economies with initial below-the-frontier per capita income and, simultaneously, relatively small (large) initial physical capital-output ratio and initial technology intensity.[2]

Although this is not the first paper to consider vertical and horizontal of R&D jointly with physical capital accumulation, to the best of our knowledge it is the first one to study transitional dynamics under a lab-equipment/fully endogenous-growth specification applied to both types of R&D. For instance, Howitt (1999, Section 6) and Zeng (2003) study endogenous growth models of vertical and horizontal R&D and physical capital accumulation, with the lab-equipment setup applied to both types of R&D, but the authors do not study the stability properties of the long-run equilibrium neither the associated transitional dynamics. Howitt (2002) and Sedgley and Elmslie (2013) build endogenous growth models with physical capital accumulation that consider a lab-equipment setup in vertical R&D, but they assume that the number of varieties grows with the population, as a result of serendipitous imitation, not deliberate (horizontal) innovation. These authors study the stability properties of the long-run equilibrium but do not analyse the characteristics of the transition paths.

Our paper is closest to Brito and Dixon (2009, 2013) and Kosempel (2004), in as much as those papers also focus on the role played by the location of the initial conditions of the dynamical system in the different regions of the phase diagram as regards the

[2]As shown later, technology intensity can also be interpreted as the human capital-technology stock ratio by means of a trivial extension of our model.

production of monotonic versus non-monotonic transition paths. Brito and Dixon (2009, 2013) focus on the behaviour of physical capital, number of firms and firm size in a Ramsey model of a stationary economy. Kosempel (2004) uses a growth model to study the dynamic behaviour of the ratios of human capital and of physical capital to the technological-knowledge stock; however, the transition path of the economic growth rate is not explicitly analysed by the authors, while growth is exogenously given in the long run equilibrium. A key contribution is also Eicher and Turnovsky (2001), who use an R&D-based growth model of the knowledge-driven type to focus on the dynamic behaviour of the growth rates of per capita output, physical capital and number of firms. However, the authors carry out their study of the transition paths by simulating a number of exogenous shocks to specific structural parameters (which then set the economy off the steady state across the phase diagram), instead of explicitly considering the set of alternative initial conditions in the phase diagram. In contrast, we look at the behaviour of all those variables both in levels and growth rates under a fully endogenous-growth framework by considering systematically the initial conditions located in the different regions of the phase diagram.A number of other papers in the endogenous-growth literature, already cited above (e.g., Gómez, 2005, Arnold, 2006, Arnold and Kornprobst, 2008, Sequeira, 2011, Growiec and Schumacher, 2013, Sequeira, Lopes, and Gomes, 2014), only focus on the existence of non-monotonic transitions that emerge as oscillatory trajectories, by analysing the effect of shifts in the values of key structural parameters on the imaginary part of the stable eigenvalues.

The remainder of the paper is as follows. In Section 2, we present the model, derive the dynamic general equilibrium and study the interior long-run equilibrium in terms of its existence, uniqueness, and local dynamics properties. In Section 4, we study the rich transitional dynamics that arise in the presence of a multi-dimensional stable manifold, by considering alternative initial conditions of the dynamical system. Section 5 discusses the results and concludes.

2. The model

We study a dynamic general equilibrium model where a single competitively-produced final good can be used in consumption, accumulation of physical capital, and vertical and horizontal R&D. The economy is populated by infinitely-lived households who inelastically supply labour to final-good firms. In turn, families make consumption decisions and invest in firms' equity. The final good is produced using labour and a continuum of intermediate goods indexed by $\omega \in [0, N]$. Potential entrants into the intermediate-good sector can devote resources either to horizontal or to vertical R&D. The former increases the number of intermediate-good varieties N, each produced by a specific industry, while the latter increases the quality of the intermediate good of an existing variety/industry, indexed by $j(\omega)$. The quality level $j(\omega)$ then impacts the final-good production by a factor $\lambda^{j(\omega)}$, where $\lambda > 1$ is a parameter measuring the size of each quality upgrade.

2.1. Production and price decisions

The final-good firm has the following production technology,

$$Y(t) = L^{1-\alpha} \cdot \int_0^{N(t)} \left[\lambda^{j(\omega,t)} \cdot X(\omega,t)\right]^\alpha d\omega,\ 0 < \alpha < 1,\ \lambda > 1, \quad (1)$$

where L is the labour input, $\lambda^{j(\omega,t)} \cdot X(\omega,t)$ is the input of intermediate good ω measured in efficiency units,[3] and $N(t)$ is the measure of varieties of these goods, all taken at time t. Final producers are price-takers in all the markets they participate. They take wages, $w(t)$, and input prices $p(\omega,t)$ as given and sell their output at a price equal to unity. From the profit maximisation conditions, we determine the demand of intermediate good ω as

$$X(\omega,t) = L \cdot \left(\frac{\lambda^{j(\omega,t)\alpha} \cdot \alpha}{p(\omega,t)}\right)^{\frac{1}{1-\alpha}},\ \omega \in [0, N(t)]. \quad (2)$$

The intermediate good is non-durable and is produced using capital, according to the production function $X(\omega,t) = K(\omega,t)$, where $K(\omega,t)$ is the input of capital. Intermediate good ω is produced with a cost function $r(t)K(\omega,t) = r(t)X(\omega,t)$, where the cost of capital is the equilibrium market real interest rate, $r(t)$.[4] The intermediate-good sector consists of a continuum $N(t)$ of industries. There is monopolistic competition if we consider the whole sector: the monopolist in industry ω fixes the price $p(\omega,t)$ but faces the isoelastic demand curve (2). Profit in industry ω is thus $\pi(\omega,t) = (p(\omega,t) - r(t)) \cdot X(\omega,t)$, and the profit maximising price is a markup over marginal cost,[5] $p(\omega,t) \equiv p(t) = r(t)/\alpha > 1$, which is constant across industries but possibly variable over time. The quality of the intermediate good ω can be characterised by the quality index $q(\omega,t) \equiv \lambda^{j(\omega,t)\frac{\alpha}{1-\alpha}}$. Then, from (2) and the mark-up, the quantity produced of ω is $X(\omega,t) = L \cdot \left(\alpha^2/r(t)\right)^{\frac{1}{1-\alpha}} \cdot q(\omega,t)$.

On the other hand, capital market equilibrium requires $K(t) = \int_0^{N(t)} K(\omega,t)d\omega = \int_0^{N(t)} X(\omega,t)d\omega = \tilde{X}(t) \cdot Q(t)$, where $\tilde{X}(t) \equiv L \cdot \left(\alpha^2/r(t)\right)^{\frac{1}{1-\alpha}}$ and

$$Q(t) = \int_0^{N(t)} q(\omega,t)d\omega, \quad (3)$$

which is the aggregate quality index. The latter measures the technological-knowledge stock of the economy, since, by assumption, there are no intersectoral spillovers. Given the capital market equilibrium condition, the quantity produced of ω can be expressed

[3] In equilibrium, only the top quality of each ω is produced and used; thus, $X(j,\omega,t) = X(\omega,t)$. Henceforth, we only use all arguments (j,ω,t) if they are useful for expositional convenience.

[4] For sake of simplicity, we assume the rate of depreciation is zero.

[5] We assume that $\frac{1}{\alpha} \leq \lambda$; i.e., if $\frac{1}{\alpha}$ is the price of the top quality, the price of the next lowest grade, $\frac{1}{\alpha\lambda}$, is less than the unit marginal cost. In this case, lower grades are unable to provide any effective competition, and the top-quality producer can charge the unconstrained monopoly price.

alternatively as $X(\omega,t) = \tilde{X}(t) \cdot q(\omega,t) = K(t)/Q(t) \cdot q(\omega,t)$. By using the two expressions for $X(\omega,t)$, we find

$$r(t) = \alpha^2 \cdot k(t)^{\alpha-1}, \tag{4}$$

where $k \equiv K/(LQ)$. This equation expresses the condition that the cost of capital must equal its marginal revenue product. By using $X(\omega,t)$ and $r(t)$, we get the optimal profit accrued by the monopolist in ω

$$\pi(\omega,t) = \pi_0 \cdot L \cdot k(t)^{\alpha} \cdot q(\omega,t), \tag{5}$$

where $\pi_0 \equiv \alpha(1-\alpha)$ is a positive constant. Also by means of $X(\omega,t)$, we get total optimal profits, total intermediate-good production, and total final-good production,

$$\Pi(t) = \int_0^{N(t)} \pi(\omega,t)d\omega = \pi_0 \cdot L \cdot k(t)^{\alpha} \cdot Q(t). \tag{6}$$

$$X(t) = \int_0^{N(t)} X(\omega,t)d\omega = k(t) \cdot L \cdot Q(t) = K(t), \tag{7}$$

$$Y(t) = L \cdot Q(t) \cdot k(t)^{\alpha}. \tag{8}$$

2.2. R&D

We consider two R&D sectors, one targeting horizontal innovation and the other vertical innovation. Each new design (a new variety or a higher quality good) is granted a patent and thus a successful innovator retains exclusive rights over the use of his/her good. Both vertical and horizontal R&D are performed by (potential) entrants, and successful R&D leads to the set-up of a new firm in either an existing or in a new industry (e.g., Howitt, 1999; Strulik, 2007; Gil, Brito, and Afonso, 2013). There is perfect competition among entrants and free entry in R&D business.

2.2.1. Vertical R&D

By improving on the current top quality level $j(\omega,t)$, a successful vertical R&D firm earns monopoly profits from selling the leading-edge input of $j(\omega,t)+1$ quality to final-good firms. A successful innovation will instantaneously increase the quality index in ω from $q(\omega,t) = q(j)$ to $q^+(\omega,t) = q(j+1) = \lambda^{\alpha/(1-\alpha)}q(\omega,t)$. In equilibrium, the lower quality good is priced out of business and the entrant replaces the incumbent monopolist, i.e., there is a creative-destruction effect.

Let $I_i(j)$ denote the Poisson arrival rate of vertical innovations (vertical-innovation rate) by potential entrant i in industry ω when the highest quality is j. Rate $I_i(j)$ is independently distributed across firms, across industries and over time, and depends on the flow of resources $R_{vi}(j)$ committed by potential entrant at time t, measured in units of the final good (e.g., Barro and Sala-i-Martin, 2004, ch. 7). Rate $I_i(j)$ features constant returns in R&D expenditures, $I_i(j) = R_{vi}(j)/\Phi(j)$, where the cost $\Phi(j)$ is homogeneous across i in industry ω. Aggregating across i in ω, we get $R_v(j) = \sum_i R_{vi}(j)$ and $I(j) = \sum_i I_i(j)$, and thus

$$I(j) = \frac{1}{\Phi(j)} R_v(j), \tag{9}$$

where $\Phi(j) = \zeta \cdot L \cdot q(j+1)$, and $\zeta > 0$ is a constant (flow) fixed cost. This equation incorporates an R&D complexity effect, implying that the larger the level of quality, q, the costlier it is to introduce a further jump in quality.[6] It also incorporates a market complexity effect, implying that an increase in market scale dilutes the effect of R&D outlays on innovation probability; this captures the idea that the difficulty of introducing new qualities and replacing old ones is proportional to the market size measured by employed labour in efficiency units and removes the undesirable scale effects on growth (e.g., Barro and Sala-i-Martin, 2004, ch. 7; Etro, 2008).

As the terminal date of each monopoly arrives as a Poisson process with frequency $I(j)$ per (infinitesimal) increment of time, the present value of a monopolist's profits is a random variable. Let $V(j)$ denote the expected value of an incumbent firm with current quality level $j(\omega, t)$,[7]

$$V(j) = \pi_0 \cdot L \cdot q(j) \int_t^{\infty} k(t)^{\alpha} \cdot e^{-\int_t^s (r(v)+I(j))dv} ds \tag{10}$$

where r is the equilibrium market real interest rate and $\pi_0 \cdot L \cdot q(j) = \pi \cdot k^{-\alpha}$, given by (5), is constant in-between innovations. Because physical capital and R&D investment both represent foregone consumption (see Subsection 2.5, below), the real rate of return to R&D is equal to that for capital, r. Free-entry prevails in vertical R&D such that the condition $I(j) \cdot V(j+1) = R_v(j)$ holds, and thus $V(j+1) = \Phi(j) = \zeta \cdot L \cdot q(j+1)$. Next, we determine $V(j+1)$ analogously to (10) and time-differentiate the resulting expression. If we also consider (5), we get the no-arbitrage condition facing a vertical innovator

$$r(t) + I(t) = \frac{\pi_0 \cdot k(t)^{\alpha}}{\zeta} \tag{11}$$

It has the implication that the rates of vertical entry are symmetric across industries, $I(\omega, t) = I(t)$.[8]

Solving equation (9) for $R_v(\omega, t) = R_v(j)$ and aggregating across industries ω, we determine total resources devoted to vertical R&D, $R_v(t) = \int_0^{N(t)} R_v(\omega, t)\, d\omega = \int_0^{N(t)} \zeta \cdot L \cdot q^{+}(\omega, t) \cdot I(\omega, t)\, d\omega$. As the innovation rate is industry independent, then

[6] The way Φ depends on j implies that the increasing difficulty of creating new qualities exactly offsets the increased rewards from marketing higher qualities – see (9) and (5). This allows for a constant vertical-innovation rate over t and across ω along the BGP, i.e., a symmetric equilibrium.

[7] We assume that entrants are risk-neutral and, thus, only care about the expected value.

[8] Observe that, from (5) and (9), we have $\frac{\dot{\pi}(\omega,t)}{\pi(\omega,t)} - \alpha\frac{\dot{k}(t)}{k(t)} = I(\omega, t) \cdot \left[\dot{j}(\omega, t) \cdot \left(\frac{\alpha}{1-\alpha}\right) \cdot \ln \lambda\right]$ and $\frac{\dot{R}_v(\omega,t)}{R_v(\omega,t)} - \frac{\dot{I}(\omega,t)}{I(\omega,t)} = I(\omega, t) \cdot \left[\dot{j}(\omega, t) \cdot \left(\frac{\alpha}{1-\alpha}\right) \cdot \ln \lambda\right]$. Then, if we time-differentiate the free-entry condition considering (10) and the equations above, we get $r(t) = \frac{\pi(j+1) \cdot I(j)}{R_v(j)} - I(j+1)$, which can then be re-written as (11).

$$R_v(t) = \zeta \cdot L \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot I(t) \cdot Q(t). \tag{12}$$

2.2.2. Horizontal R&D

Variety expansion arises from R&D aimed at creating a new intermediate good. Under perfect competition among R&D firms and constant returns to scale at the firm level, instantaneous entry is obtained as $\dot{N}_e(t) = R_{ne}(t)/\eta(t)$, where $\dot{N}_e(t)$ is the contribution to the instantaneous flow of the new good by potential entrant e at a cost of $\eta(t)$ and $R_{ne}(t)$ is the flow of resources devoted to horizontal R&D by e at time t, measured in units of the final good (e.g., Barro and Sala-i-Martin, 2004, ch. 6). The cost η is assumed to be symmetric. Then, $R_n = \sum_e R_{ne}$ and $\dot{N}(t) = \sum_e \dot{N}_e(t)$, implying

$$\dot{N}(t) = \frac{1}{\eta(t)} R_n(t). \tag{13}$$

Following Gil, Brito, and Afonso (2013), we assume that the cost of horizontal entry is increasing in both the number of existing varieties, N, and the number of new entrants, $\dot{N}$, $\eta(t) = \phi \cdot N(t)^\sigma \cdot \dot{N}(t)^\gamma$, where $\phi > 0$ is a constant (flow) fixed cost, and $\sigma, \gamma > 0$. This equation introduces two types of decreasing returns associated to horizontal R&D. Dynamic decreasing returns to scale are modeled by the dependence of η on N and capture an R&D complexity effect: the larger the number of existing varieties, the costlier it is to introduce new varieties. The dependence of η on $\dot{N}$ means that the entry technology displays static decreasing returns to scale at the aggregate level, due to, e.g., congestion effects reflecting the physical nature of this type of R&D (in contrast with the more immaterial nature of vertical R&D). We assume these to be entirely external to the firm. An implication is that new varieties are brought to the market gradually, instead of through a lumpy adjustment. This is in line with the stylised facts on entry (e.g., Geroski, 1995), according to which entry occurs mostly at small scale since adjustment costs penalise large-scale entry. The existing growth literature of horizontal R&D deals with the two features separately: some models only display dynamic decreasing returns (e.g., Evans, Honkapohja, and Romer, 1998; Barro and Sala-i-Martin, 2004, ch. 6), while others only assume static decreasing returns (e.g., Arnold, 1998; Howitt, 1999; Jones and Williams, 2000).

The innovator enters with a new intermediate good whose quality level is drawn randomly from the distribution of existing varieties (e.g., Howitt, 1999). Thus, the expected quality level of the horizontal innovator is $\bar{q}(t) = \int_0^{N(t)} q(\omega, t) d\omega / N(t) = Q(t)/N(t)$. As monopoly power will be also terminated by the arrival of a successful vertical innovator in the future, the benefits from entry are given by

$$V(\bar{q}) = \pi_0 \cdot L \cdot \bar{q}(t) \int_t^\infty k(t)^\alpha \cdot e^{-\int_t^s [r(\nu) + I(\bar{q})] d\nu} ds, \tag{14}$$

where $\pi_0 L\bar{q} = \bar{\pi}k^{-\alpha}$. The free-entry condition is now $\dot{N} \cdot V(\bar{q}) = R_n$, which simplifies to $V(\bar{q}) = \eta(t)$. Using (14) and time-differentiating the resulting expression, yields the no-arbitrage condition facing a horizontal innovator

$$r(t) + I(t) = \frac{\bar{\pi}(t)}{\eta(t)}. \tag{15}$$

Total resources devoted to horizontal R&D are given by $R_n(t) = \eta(t)\dot{N}(t)$, from (13).

2.2.3. Inter-R&D no-arbitrage condition

Before deciding which type of R&D to perform, the potential entrant should evaluate the best type of entry. At the margin, she/he should be indifferent between the two types. If we equate the effective rate of return $r + I$ for both types of entry by considering (11) and (15), we get the no-arbitrage condition

$$\bar{q}(t) = \frac{Q(t)}{N(t)} = \frac{\eta(t)}{\zeta \cdot L}, \tag{16}$$

which equates the cost of horizontal R&D, η, to the average cost of vertical R&D, $\bar{q}\zeta L$.

Equation (16) can be equivalently recast as

$$\dot{N}(t) = x(Q(t), N(t)) \cdot N(t), \tag{17}$$

where

$$x(Q, N) = \left(\frac{\zeta \cdot L}{\phi}\right)^{\frac{1}{\gamma}} \cdot Q^{\frac{1}{\gamma}} \cdot N^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)}, \tag{18}$$

which expresses a channel between vertical innovation and firm dynamics. It shows that the horizontal-entry rate, $\dot{N}/N$, depends negatively on N and positively on Q: the first relationship results from the complexity effect incorporated in equation (13), and the second is an implication of the complementarity between the horizontal-entry rate and the technological-knowledge stock, which in turn comprises both the horizontal and the vertical-innovation components, $Q = N \cdot \bar{q}$. By time-differentiating $\bar{q} = Q/N$ using (3), we see that the rate of growth of average quality is equal to the expected arrival rate of a vertical innovation multiplied by the quality shift it introduces: $\dot{\bar{q}}(t)/\bar{q}(t) = \Xi \cdot I(t)$, where both the quality shift, $\Xi \equiv (q^+ - q)/q = \left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)$ (which captures the creative-destruction effect), and the vertical-innovation rate, I, are industry-independent. Then, using (17), we get

$$\dot{Q}(t) = (I(t) \cdot \Xi + x(Q(t), N(t))) \cdot Q(t), \tag{19}$$

The vertical-innovation rate is endogenous and will be determined as an economy-wide function below. Equation (19) introduces a second dynamic interaction between the two types of entry, in this case between the number of varieties, N, and the rate of growth of the quality index of the economy.

2.3. Households

The economy is populated by a constant number of infinitely-lived households who consume and earn income from investments in financial assets (equity) and from labour. Households inelastically supply labour to final-good firms; thus, total labour supply, L, is exogenous and constant. Consumers have perfect foresight regarding the technological change over time and choose the path of aggregate consumption $\{C(t), t \geq 0\}$ to maximise discounted lifetime utility

$$U = \int_0^{\infty} \left(\frac{C(t)^{1-\theta} - 1}{1-\theta} \right) \cdot e^{-\rho t} dt, \tag{20}$$

where $\rho > 0$ is the subjective discount rate and $\theta > 0$ is the inverse of the intertemporal elasticity of substitution in consumption, subject to the flow budget constraint

$$\dot{a}(t) = r(t) \cdot a(t) + w(t) \cdot L - C(t), \tag{21}$$

where a denotes households' real financial assets holdings and its initial level, $a(0)$, is given. The optimal path of consumption (Euler equation) and the transversality condition are

$$\dot{C}(t) = \frac{1}{\theta} \cdot (r(t) - \rho)\, C(t) \tag{22}$$

$$\lim_{t \to +\infty} e^{-\rho t} \cdot C(t)^{-\theta} \cdot a(t) = 0 \tag{23}$$

2.4. Macroeconomic aggregation

The aggregate financial wealth held by all households is $a(t) = K(t) + \int_0^{N(t)} V(\omega, t) d\omega$, which, from the arbitrage condition between vertical and horizontal entry, yields $a(t) = K(t) + \eta(t) \cdot N(t)$. Taking time derivatives and comparing with (21), we get an expression for the aggregate flow budget constraint which is equivalent to the product market equilibrium condition (see Appendix A)

$$Y(t) = C(t) + \dot{K}(t) + R_v(t) + R_n(t) \tag{24}$$

If we substitute the expressions for the aggregate output (8) and for total R&D expenditures (12) and (13), and then solve for $\dot{K}$ using (16) and (17), we get the endogenous rate of physical-capital accumulation

$$\dot{K}(t) = L \cdot Q(t) \cdot \left(k\,(t)^{\alpha} - \frac{C(t)}{L \cdot Q(t)} - \zeta \cdot x(Q(t), N(t)) - I(t) \cdot \zeta \cdot \lambda^{\frac{\alpha}{1-\alpha}} \right) \tag{25}$$

In parallel, recall (4) and (11), to get $r(t) \equiv r(Q, K)$ and hence the endogenous vertical-innovation rate,

$$I(Q,K) = \frac{\pi_0 \cdot k(t)^{\alpha}}{\zeta} - r(Q,K), \tag{26}$$

as a function which is decreasing in the aggregate quality level and increasing in the physical-capital stock. As function $I(Q,K)$ may be negative, the relevant innovation rate at the macroeconomic level is $I^{+}(Q,K) = \max\{I(Q,K),0\}$. We emphasise the complementarity between vertical innovation and physical-capital accumulation, here made clear by the fact that, if K is too low, vertical R&D shuts down (i.e., $I^{+}(\cdot) = 0$).

2.5. Dynamic general equilibrium

The dynamic general equilibrium is defined by the allocation $\{X(\omega,t), \omega \in [0,N(t)], t \geq 0\}$, by the prices $\{p(\omega,t), \omega \in [0,N(t)], t \geq 0\}$ and by the aggregate paths $\{C(t), N(t), Q(t), K(t), I(t), r(t), t \geq 0\}$, such that: (i) consumers, final-good firms and intermediate-good firms solve their problems; (ii) vertical, horizontal and consistency free-entry conditions are met; and (iii) markets clear. We focus on the region of the state space where $I^{+}(\cdot) = I(\cdot) > 0$, such that the equilibrium paths can be obtained from the system

$$\dot{C} = \frac{1}{\theta} \cdot (r(Q,K) - \rho) \cdot C \tag{27}$$

$$\dot{Q} = (I(Q,K) \cdot \Xi + x(Q,N)) \cdot Q \tag{28}$$

$$\dot{K} = LQ \cdot \left(k^{\alpha} - \frac{C}{LQ} - \zeta \cdot x(Q,N) - I(Q,K) \cdot \zeta \cdot \lambda^{\frac{\alpha}{1-\alpha}} \right) \tag{29}$$

$$\dot{N} = x(Q,N) \cdot N \tag{30}$$

given $K(0)$, $Q(0)$ and $N(0)$, and the transversality condition (23), which may be re-written as

$$\lim_{t \to +\infty} e^{-\rho t} \cdot C(t)^{-\theta} \cdot (K(t) + \zeta \cdot L \cdot Q(t)) = 0 \tag{31}$$

3. Equilibrium dynamics

3.1. Balanced-growth path

Let $g_y \equiv \dot{y}/y$ denote the growth rate of variable $y(t)$. As the functions in system (27)-(29) are homogeneous, a balanced-growth path (BGP) may exist if the further necessary conditions are verified: (i) the asymptotic growth rates of consumption, of physical capital and of technological knowledge are constant and equal, $g_C = g_K = g_{\dot{K}} = g_Q = g$; (ii) the vertical-innovation rate and the real interest rate are asymptotically trendless, $g_I = g_r = 0$; and (iii) the asymptotic growth rates of the quality index and the number of varieties are monotonically related, $g_Q = (\sigma + \gamma + 1) \cdot g_N$. Observe, from (18), that $x = g_N$ is always positive if $N > 0$.

Under these conditions, recall (18) and $k(t) \equiv K(t)/(L \cdot Q(t))$, and let $z(t) \equiv C(t)/(L \cdot Q(t))$, with the property that, along the BGP, $\dot{x} = \dot{z} = \dot{k} = 0$. Then, it is easily shown that the system (27)-(29) is equivalent to

$$\dot{x}(t) = \left[I(k(t)) \cdot \Xi \cdot \frac{1}{\gamma} - \left(\frac{\sigma}{\gamma} + 1 \right) \cdot x(t) \right] \cdot x(t) \tag{32}$$

$$\dot{z}(t) = \left[\frac{1}{\theta} \cdot (r(k(t)) - \rho) - \Xi \cdot I(k(t)) - x(t) \right] \cdot z(t) \tag{33}$$

$$\dot{k}(t) = \left[\frac{1}{k(t)} \cdot \left(k(t)^{\alpha} - z(t) - \zeta \cdot x(t) - \zeta \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot I(k(t)) \right) - \Xi \cdot I(k(t)) - x(t) \right] \cdot k(t) \tag{34}$$

where $r(k) \equiv r(Q, K) > 0$ and $I(k) \equiv I(Q, K) > 0$ (see (4) and (11)). Observe that in this system of equations there is one jump-like variable, z, and two state-like (predetermined) variables, x and k. Then, this system, plus the transversality condition and the initial conditions $x(0)$ and $k(0)$, describe the transitional dynamics and the BGP, by jointly determining $(x(t), z(t), k(t))$. Given the latter, we can determine the original variables $N(t)$, $K(t)$, $\dot{K}(t)$ and $C(t)$, for a given $Q(t)$.

The long-run equilibrium values of x, k and z (steady-state values) are obtained by setting the right-hand side of each of the previous equations to zero. The dynamics for $\dot{k}$ provides the steady-state value of the variable $z(t)$. In fact, solving for the right-hand side of equation (34) equal to zero yields

$$z^* = (k^*)^{\alpha} - (\zeta + k^*) \cdot x^* - \zeta \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot I(k^*) - k^* \cdot \Xi \cdot I(k^*), \tag{35}$$

where the superscript $*$ denotes the steady-state value of a variable. Generically, the steady-state values of x and k are defined only implicitly. These values solve the following two equations, obtained from equations (32) and (33) by assuming $z(t) > 0$ and using (4) and (11),

$$x = f_1(k) = \frac{\alpha^2 k^{\alpha-1} - \rho}{\theta(\sigma + \gamma + 1)} \tag{36}$$

$$x = f_2(k) = \frac{\Xi}{(\sigma + \gamma)\zeta} \left(\pi_0 k^{\alpha} - \zeta \alpha^2 k^{\alpha-1} \right). \tag{37}$$

The system above defines the pair (x^*, k^*), which, jointly with z^* in (35), represents a steady-state equilibrium with balanced growth in the usual sense, such that the endogenous growth rates are constant and positive, $g_N^* = x^* > 0$ and $g_Q^* = g^* = (\sigma + \gamma + 1)x^* > 0$. Observe also that, since the economic growth rate is $g_Y = g_Q(t) + \alpha g_k(t)$ (see (8)), then $g_Y^* = g_Q^* = g^*$. Thus, our model predicts, under a sufficiently productive technology, a BGP with a constant positive economic growth rate, where $g^* > g_N^*$ due to the growth of intermediate-good quality, $\Xi \cdot I^*$ (see (19)).

The next proposition establishes that a meaningful steady-state exists.

Proposition 1. Assuming $\rho < \alpha^2 \left(\frac{1-\alpha}{\zeta\alpha} \right)^{1-\alpha}$, the model possesses at least one interior steady-state satisfying $k \in]\bar{k}_{min}; \bar{k}_{max}[$ with:

$$\bar{k}_{min} = \frac{\zeta\alpha}{1-\alpha} \quad ; \quad \bar{k}_{max} = \left(\frac{\alpha^2}{\rho}\right)^{\frac{1}{1-\alpha}}.$$

Proof. It is clear that, once the steady-state values of x and k have been found, direct substitution in equation (35) provides the steady-state value of z. The steady-state values of x and k occur at the intersection of equations (36) and (37). We show that such an intersection occurs by noting that:

- $f_1(k)$ is a decreasing function of k, while $f_2(k)$ is increasing. In fact, since $\alpha \in (0, 1)$,

$$\begin{aligned} \frac{df_1}{dk} &= \frac{1}{\theta(\sigma+\gamma+1)}\alpha^2(\alpha-1)k^{\alpha-2} < 0 \\ \frac{df_2}{dk} &= \frac{\Xi}{(\sigma+\gamma)\zeta}k^{\alpha-2}\left[\pi_0\alpha k - \zeta\alpha^2(\alpha-1)\right] > 0. \end{aligned}$$

- The limiting behaviour of f_1 and f_2 is given by

$$\lim_{k\to+\infty} f_1(k) = -\left[\frac{\rho}{\theta(\sigma+\gamma+1)}\right] < 0 \quad ; \quad \lim_{k\to+\infty} f_2(k) = +\infty;$$

$$\lim_{k\to 0^+} f_1(k) = +\infty \quad ; \quad \lim_{k\to 0^+} f_2(k) = -\infty.$$

It remains to show that this intersection occurs for $x > 0$, which is the case provided f_1 and f_2 are simultaneously positive. Let $\bar{k}_{min}$ be such that $f_2(\bar{k}_{min}) = 0$. We have

$$f_2(k) = 0 \Leftrightarrow \bar{k}_{min} = \frac{\zeta\alpha}{1-\alpha}.$$

and, given the hypothesis,

$$f_1(\bar{k}_{min}) = \frac{\alpha^2\left(\frac{\zeta\alpha}{1-\alpha}\right)^{\alpha-1} - \rho}{\theta(\sigma+\gamma+1)} > 0.$$

The interval of values of k for which a meaningful equilibrium occurs is bounded above by the value of k for which f_1 changes sign. Let this value be $\bar{k}_{max}$. Then

$$f_1(\bar{k}_{max}) = 0 \Leftrightarrow \bar{k}_{max} = \left(\frac{\rho}{\alpha^2}\right)^{\frac{1}{\alpha-1}}.$$

□

For the particular but generically accepted case of $\alpha = 1/3$, uniqueness of the equilibrium is also guaranteed.

Lemma 1. The steady state exists, is unique and explicitly determined in a neighbourhood of $\alpha = 1/3$ and

$$\zeta > \left(\frac{27}{4}\right) \frac{d}{(\sigma+\gamma)^3}(d+\sigma+\gamma), \tag{38}$$

where $d = \Xi\theta(\sigma+\gamma+1)$.

Proof. The equation $f_1(k) = f_2(k)$ is equivalent to

$$f_1(k) = f_2(k) \Leftrightarrow k^\alpha + bk^{\alpha-1} + a = 0 \Leftrightarrow k + ak^{1-\alpha} + b = 0$$

where

$$a = \frac{(\sigma+\gamma)\zeta\rho}{\Xi\theta(\sigma+\gamma+1)\pi_0} > 0 \quad ; \quad b = -\frac{\alpha\left[\Xi\theta(\sigma+\gamma+1)+\sigma+\gamma\right]}{\Xi\theta(\sigma+\gamma+1)(1-\alpha)} < 0.$$

By writing $1-\alpha = (m-n)/m$ and $k = y^m$, the previous polynomial becomes

$$y^m + ay^{m-n} + b = 0,$$

which is of degree 3 when $\alpha = 1/3$. Conditions for existence and uniqueness of real zeros of degree 3 polynomials are available by Cardano-Tartaglia's method and in the present case amount to

$$\frac{a^3 b}{27} + \frac{b^2}{4} < 0 \Leftrightarrow b\left(\frac{a^3}{27} + \frac{b}{4}\right) < 0 \Rightarrow_{(b<0)} \frac{a^3}{27} + \frac{b}{4} > 0,$$

leading to inequality (38). Notice that continuity of the equilibrium guarantees uniqueness in an open set containing $\alpha = 1/3$. □

As in Howitt and Aghion (1998), capital accumulation, such that k is constant in the long-run equilibrium, is necessary for growth to be sustained. Since capital is assumed to be a necessary factor of production, $k \to 0$ would drive output, and hence profit, to zero (see (5) and (8)), while the interest rate would be driven to infinity (see (4)). Thus, the reward to innovation would be driven to zero, making it impossible for the conditions (11) and (15) to be satisfied. The reverse is also true, since, without innovation, diminishing returns would eventually put capital accumulation to a halt. Hence, capital accumulation and innovation are complementary mechanisms.

3.2. Aggregate transitional dynamics

Next, we qualitatively characterise the local dynamics properties in a neighbourhood of the long-run equilibrium, by studying the solution of the linearised system obtained from (32)-(34). The Jacobian matrix at the steady-state values (x^*, k^*, z^*) is given by

$$J = \begin{pmatrix} -\left(\frac{\sigma}{\gamma}+1\right)x^* & 0 & J_{13}(x^*,k^*) \\ -z^* & 0 & J_{23}(x^*,z^*) \\ -\zeta - k^* & -1 & J_{33}(x^*,k^*) \end{pmatrix}$$

where

$$J_{13} = B_0 \frac{\Xi}{\gamma} \left(\frac{k^*}{\zeta} + 1 \right) (k^*)^{\alpha-2} x^* > 0,$$

$$J_{23} = -B_0 \left(\frac{1}{\theta} + \Xi + \Xi \frac{k^*}{\zeta} \right) (k^*)^{\alpha-2} z^* < 0,$$

$$J_{33} = -B_1 (k^*)^{\alpha} + B_2 (k^*)^{\alpha-1} - B_3 (k^*)^{\alpha-2} - x^*,$$

with $B_0 = \alpha^2(1-\alpha)$; $B_1 = (1+\alpha)\Xi B_0/(\zeta\alpha)$, $B_2 = \alpha\left[1 - \lambda^{\frac{\alpha}{1-\alpha}}\alpha(1-2\alpha) - \alpha^2\right]$ and $B_3 = \zeta\lambda^{\frac{\alpha}{1-\alpha}} B_0$. Notice that J_{33} may have either sign. Standard calculations produce the values for the trace and determinant of J,

$$\mathrm{tr}(J) = -\left(\frac{\sigma}{\gamma} + 1\right) x^* + J_{33} \quad \text{and} \quad \det(J) = z^* J_{13} - \left(\frac{\sigma}{\gamma} + 1\right) x^* J_{23}.$$

The characteristic equation for J is then given by $q^3 - \mathrm{tr}(J) q^2 + a_2 q - \det(J)$, where $a_2 = q_1 q_2 + q_1 q_3 + q_2 q_3$ and q_i is an eigenvalue of J.

Since there are two state-like variables, x and k, and one jump-like variable, z, local saddle-path stability requires that J has two eigenvalues with negative real parts and one with positive real part. According to the Routh-Hurwitz Theorem, the number of eigenvalues of J with positive real parts equals the number of changes of sign in the following sequence

$$-1, \quad \mathrm{tr}(J), \quad -a_2 + \frac{\det(J)}{\mathrm{tr}(J)}, \quad \det(J).$$

It is clear that $\det(J) > 0$. If $\mathrm{tr}(J) < 0$ then, regardless of the sign of the third element in the sequence, there is exactly one change of sign in the sequence and we have a two-dimensional stable manifold at the equilibrium (i.e., a two-dimensional saddle path). When $\mathrm{tr}(J) > 0$, we have one, and only one, change of sign if and only if $-a_2 + \det(J)/\mathrm{tr}(J) > 0$. Note that here $a_2 < 0$ is a sufficient (not necessary) condition for a unique sign change.

Next, we establish sufficient conditions for the existence of a two-dimensional saddle path. Although these are not the most generic conditions possible, we remark that:

- They require conditions only on parameters α and λ, with the remaining parameters being not constrained;
- Continuity of the eigenvalues with α and λ ensures that those conditions remain valid in an open set containing the specified parameter values.

Therefore, even though our numerical ilustrations of the dynamics (see Section 4, below) require fixed values for the parameters, they are qualitatively equivalent in an open set of nearby parameter values. This open set contains all possible values for all parameters except α and λ.

Furthermore, the condition $\det(J) > 0$, provided the eigenvalues are real, ensures hyperbolicity of the equilibrium. Therefore, a linear approximation of the dynamics is guaranteed to be a good approximation. We use this fact to simplify our subsequent study of the dynamics.

Proposition 2. There exists a two-dimensional saddle path in an open set of the parameter space containing $\alpha = 1/3$ and such that $\lambda > (92/63)^2$.

Proof. From $\mathrm{tr}(J) = -(\sigma/\gamma + 1)\, x^* + J_{33}$ it suffices to show that

$$J_{33} = -B_1(k^*)^\alpha + B_2(k^*)^{\alpha-1} - B_3(k^*)^{\alpha-2} - x^* < 0.$$

The last term is clearly negative so it suffices to show that the remaining three terms also add up to a negative number. We note that

$$-B_1(k^*)^\alpha + B_2(k^*)^{\alpha-1} - B_3(k^*)^{\alpha-2} < 0 \Leftrightarrow -B_3 + B_2 k^* - B_1 k^{*2} < 0.$$

This occurs trivially when the polynomial of degree two in k^* has no real roots. No real roots appear when

$$B_2^2 - 4B_1B_3 < 0.$$

For $\alpha = 1/3$, we obtain

$$B_2^2 - 4B_1B_3 = \frac{A^2}{27^2}(-63X^2 + 48X + 64), \;\; X = \sqrt{\lambda},$$

which is negative for

$$X > \frac{24 + \sqrt{4608}}{63}.$$

We have

$$\frac{24 + \sqrt{4608}}{63} < \frac{24 + 68}{63} = \frac{92}{63}$$

and, hence, $\lambda > (92/63)^2 \simeq 2.132$ guarantees that $\mathrm{tr}(J) < 0$. □

It follows that, under the above sufficient conditions, the long-run equilibrium is "well-behaved", meaning that instability, local indeterminacy and limit cycles cannot occur.

4. Transition paths under distinct initial conditions

In order to analyse the transitional dynamics, we perform a numerical illustration. We consider the following set of baseline parameter values: $\gamma = 1.2$, $\sigma = 1.2$, $\phi = 1$, $\zeta = 3.75$, $\lambda = 3$, $\rho = 0.02$, $\theta = 1.5$, $\alpha = 1/3$, and $L = 1$. Given that, along the BGP, $g_Q - g_N = (\sigma + \gamma)g_N$, the choice of values for σ and γ is such that $(\sigma + \gamma) = 2.4$, which is the ratio between the growth rate of the average firm size and the growth rate of the number of firms we have found in the empirical data.[9] The values for λ, θ, ρ and α were

[9] The data, which are available from the authors upon request, concern 23 European countries in the period $1995 - 2007$ and were taken from the Eurostat online database, at `http://epp.eurostat.ec.europa.eu`.

x^*	k^*	z^*	g^*	μ_1	μ_2	μ_3
0.0074	2.6634	1.1330	0.0252	−0.2064	−0.0192	0.1499

Table 1: Steady-state values of selected endogenous variables and of the eigenvalues ($\mu_j, i = 1, 2, 3$) for the baseline parameter values.

set in line with the standard growth literature (see, e.g., Barro and Sala-i-Martin, 2004), whereas the normalization of L to unity at every t implies that all aggregate magnitudes can be interpreted as per capita magnitudes. The values of the remaining parameters were chosen to calibrate the BGP aggregate growth rate around 2.5%/year.

For these parameter values, the analytical conditions derived for the existence and uniqueness of the long-run equilibrium and for the existence of a two-dimensional stable manifold are verified. The equilibrium is hyperbolic, which garantees that we can make use of the Hartman Grobman Theorem to study the dynamics by linearisation. Table 1 shows the steady-state values and eigenvalues generated by the baseline parameter values, which confirm that the stable manifold results from the existence of two negative real eigenvalues.

[Table 1 goes about here]

We performed a numerical exercise and concluded that complex eigenvalues, resulting in oscillatory behaviour towards the steady state, are likely to emerge only for implausible values of the parameters. Additionally, in case of oscillatory behaviour it is imperceptible and does not change the economic interpretation of the transitional dynamics.

Remark. Complex eigenvalues can emerge for different parameter values. For instance, if $\sigma = \gamma = 0.2$ (which are implausibly low values for these parameters) and $\zeta = 4.6$, then the eigenvalues would be:

$$\mu_1 = -0.1275 + 0.0576i \ \ , \mu_2 = -0.1275 - 0.0576i \ \ , \mu_3 = 0.1426,$$

with complex eigenvalues associated to the stability. This oscillatory behaviour results from a change in the entry cost on both margins: a decrease in the extensive and an increase in the intensive margin, thus illustrating the tension between the horizontal and vertical barriers to entry.

Because we wish to focus on the behaviour of the state-like variables, x and k, we will abstract from the jump-like variable, z. With that purpose, we follow a similar approach to, e.g., Eicher and Turnovsky (2001) and project the stable space of the linearised system onto the horizontal plane (x, k).[10] Thus, henceforth, we are going to consider the projected dynamics. In the (x, k)-space, the dynamics can be described by the following second-order system

[10] Since this consists in a non-orthogonal projection, the jump-like variable z continues to have an effect on the observed dynamics of x and k.

$$\begin{pmatrix} \dot{x} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \dfrac{\mu_1 v_{2,2} v_{1,1} - \mu_2 v_{2,1} v_{1,2}}{v_{2,2} v_{1,1} - v_{2,1} v_{1,2}} & \dfrac{(\mu_2 - \mu_1) v_{1,2} v_{1,1}}{v_{2,2} v_{1,1} - v_{2,1} v_{1,2}} \\ \dfrac{(\mu_1 - \mu_2) v_{2,1} v_{2,2}}{v_{2,2} v_{1,1} - v_{2,1} v_{1,2}} & \dfrac{\mu_2 v_{2,2} v_{1,1} - \mu_1 v_{2,1} v_{1,2}}{v_{2,2} v_{1,1} - v_{2,1} v_{1,2}} \end{pmatrix} \cdot \begin{pmatrix} x - x^* \\ k - k^* \end{pmatrix},$$

where μ_j, $i = 1, 2$, are the negative eigenvalues and $v_{i,j}$ is the ith element of the eigenvector associated to μ_j.

From (17)-(18), x can be interpreted either as the growth rate of the number of varieties (horizontal innovation rate) or, alternatively, as a measure of the intensity of use of technological knowledge (or 'technology intensity'), since x is defined as a (non-linear) transformation of Q/N , which in turn measures the proportion of the intensive vis-à-vis the extensive component of a given technological-knowledge stock. On the other hand, recalling (8) and $k \equiv K/(L \cdot Q)$, we can rewrite the latter as $k = (K/Y)^{\frac{1}{1-\alpha}}$; thus, this variable can be interpreted as a (non-linear) measure of the physical capital-output ratio.

Figure 2 illustrates the projection of the phase diagram (originally on the stable space) onto the plane (x, k). The stable space is $E^s = E_{\mu_1} + E_{\mu_2}$ where:

$$E_{\mu_1} = \{(x(t), k(t)) : k(t) - k^* = \frac{v_{2,1}}{v_{1,1}}(x(t) - x^*)\},$$
$$E_{\mu_2} = \{(x(t), k(t)) : k(t) - k^* = \frac{v_{2,2}}{v_{1,2}}(x(t) - x^*)\},$$

and E_{μ_1} and E_{μ_2} divide the phase diagram in the phases A, B, C and D. Additionally, we divide phase A and B into two separate regions (by the dashed vertical line), since they represent different economic scenarios. The regions to the left (right) of the vertical line can be seen as scenarios where the initial intensity of use of technological knowledge, $x(0)$, is relatively small (large) whereas the regions above (below) E_{μ_2} represent scenarios of relatively large (small) physical capital-ouput ratio, $k(0)$. Various transition paths are illustrated in colours, indicating the distinct possibilities of convergence. These are qualitatively different according to the region to which the set of initial conditions belongs in the plane (x, k). If we start in region C or D, both the capital-output ratio and the technology-intensity variable converge monotonically, which implies also a monotonic behaviour of the economic growth rate, as we will see below. If we start in any other region, both state-like variables have a non-monotonic transition that transmits to the economic growth rate.[11]

[Figure 2 goes about here]

We seek to provide economic interpretation for the dynamics of the state-like variables, x and k, and other key variables of interest such as: the vertical innovation rate, $I(t)$; the

[11] The transitions in regions A2 and B2 are qualitatively identical to the ones in A1 and B1, respectively. Therefore, in what follows, we are only interested in explicitly analysing the transitional dynamics in regions A1, B1, C and D.

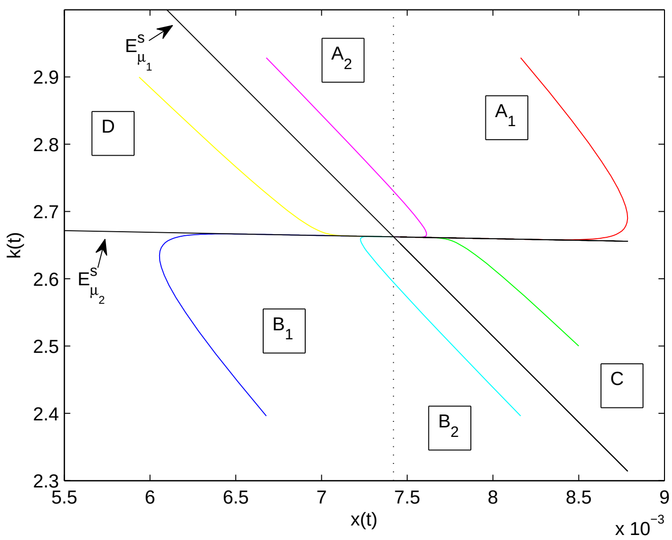

Figure 2: Projection of the phase diagram onto the plane (x, k). Various possible transitions are illustrated in colours, one for each region. In regions C and D, the convergence is monotonic, whereas in the other regions the convergence of both state-like variables is non monotonic.

economic growth rate, $g_Y(t) = I(t) \cdot \Xi + x(t) + \alpha g_k(t)$ (see (8) and (19)); the technological-knowledge stock, $Q(t)$; the stock of physical capital, $K(t)$; the number of firms/varieties, $N(t)$; average firm size, $K(t)/N(t)$; and the saving rate, $s(t) = 1 - z(t)/k(t)^{\alpha}$.Some of these variables (Q, K, N, and K/N) are non stationary in the long run and, thus, for convenience, we will represent them after being adjusted for the respective time trend: $Q_d(t) = Q(t) \cdot e^{-g_Q^* t}$; $K_d(t) = K(t) \cdot e^{-g_K^* t}$; $N_d(t) = N(t) \cdot e^{-g_N^* t}$; $(K/N)_d(t) = K_d(t)/N_d(t)$. We are also interested in the time-path of the speed of convergence of x and k, computed as $\omega(t) = -\dot{y}(t)/\left(y(t) - y^*\right)$, where y^* is the steady-state value of $y \in \{x, k\}$(see, e.g., Eicher and Turnovsky, 2001).

The initial conditions considered for each one of the different regions are such that the vertical innovation rate and the number of varieties are positive throughout the transition, while the linearisation holds as a good approximation for the dynamics.[12] In order to relate our results with the empirical evidence on transition economies, we will focus on the case of economies all of which exhibit an initial per capita output, $Y(0)/L = Q(0)^{1-\alpha}[K(0)/L]^{\alpha}$, below that of the frontier countries,[13] but yet may feature distinct combinations of $Q(0)$, $K(0)$ and $N(0)$ (and hence of $k(0)$ and $x(0)$, as referred to above).

The economic interpretation of the mechanism underlying the transitional path is as

[12] We arbitrarely consider $Q(0) = 1$ and then compute $K(0)$ and $N(0)$ by the respective relationship with $Q(0)$.

[13] We also assume that both $Q(0)$ and $K(0)/L$ are below the frontier levels.

follows, analysed counter-clockwise through the relevant regions of Figure 2.

Region A1 The initial conditions are such that $k^* < k(0)$ and $x^* < x(0)$. This would be the case of an economy characterised by relatively large initial capital-output ratio and intensity of use of technological knowledge/horizontal innovation rate. The adjustment of the economy towards the steady state is characterised by several variables following non-monotonic transition paths (see Figure 3).

The initial increase in x is justified by the complementarity between the two types of innovation, i.e., the initially high value of the vertical innovation rate, I, generates an increase in the technological-knowledge stock, Q, that stimulates horizontal innovation (see (18)). However, this effect is dampened by the rise of the horizontal entry cost, η, driven by the rapid increase in the number of varieties, N, itself a result of the large initial x. Eventually, the rise of η dominates and x follows a downward path converging to the steady state. This non-monotonic behavior is reflected in the dynamics of the corresponding speed of convergence, ω_x. Observe that the high initial I results from the existing complementarity between the capital-output ratio and the intensive margin of innovation: a larger k induces a larger profit per firm (see (5)), which generates an higher incentive to vertical innovation by rising the effective rate of return, $r + I$ (see (11)).

The physical capital-output ratio, k, is initially high and rapidly decreasing due to the low marginal productivity of physical capital, K, and the relatively large profitability of innovation activities. However, over time, the reduction in the horizontal innovation rate frees resources for the accumulation of K; eventually, the former effect is more than compensated by the latter and k rises towards the steady state. Thus, the observed undershooting of k mirrors the hump-shaped trajectory of the horizontal innovation rate, x. The speed of convergence, ω_k, reflects the non-monotonic transition of k, in particular the undershooting, which implies an infinite speed of convergence at that point. The vertical innovation rate, I, follows a similar path to k, as a consequence of the referred to complementarity between physical capital accumulation and the intensive margin of innovation.

The number of varieties, N, grows over transition at a greater rate than in the BGP (given that $x(t) > x^*$ for every finite t) and, thus, the detrended number of varieties, N_d, increases towards the steady state. In contrast, physical capital, K, starts by growing less than in the BGP but eventually accelerates to grow above the BGP – but by less than N, since the negative or very small positive g_k puts a drag on the overall growth rate of K (recall that K grows at the rate $g_k + g_Q$). Thus, detrended physical capital, K_d, observes a slight fall in the first stage of its transition path and then increases but at a slow rate towards the steady state.[14] This leads to a monotonic convex downward path of the detrended firm size, $(K/N)_d$.

The saving rate, s, begins by rising dramatically as the resources are re-allocated to horizontal R&D and while vertical R&D (reflecting the vertical innovation rate, I) is still high. It experiences a slight overshoot after which it starts falling as the decline of resources devoted to horizontal R&D more than compensates first for the recovery of the

[14]Since this is a very moderate effect, it is not noticeable in the K panel in Figure 3.

physical capital accumulation rate, g_k, and then for the increase in vertical R&D due to the recovery of I.

Finally, the economic growth rate, g_Y, is initially low, because g_k is negative, but experiences a non-monotonic, hump-shaped, transition. As g_k becomes less negative, the horizontal innovation rate, x, increases and the vertical innovation rate, I, is still high (although falling), g_Y observes a notorious increase and overshoots. However, the ensuing decline of x and g_k outweights the recovery of I and generates a gradual downward movement of g_Y towards the long-run equilibrium.

[Figure 3 goes about here]

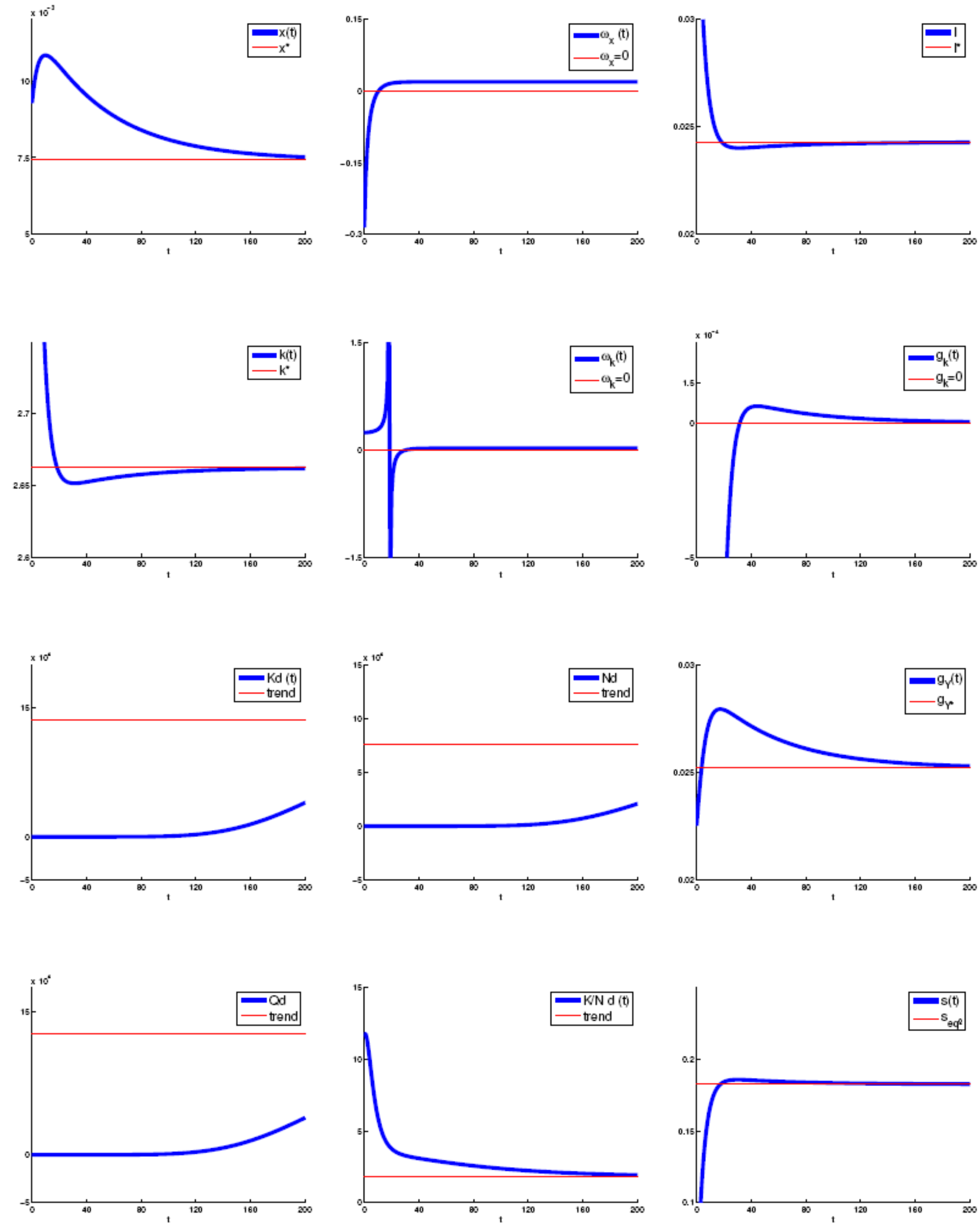

Figura 3: Transitional dynamics of the key economic variables and of the speeds of convergence of k and x (ω_k and ω_x) for an economy that sets off with $k^* < k(0)$ and $x^* < x(0)$ (region A1 of Figure 2). The economic growth rate, g_Y, and the saving rate, s, display a hump-shaped behaviour, while detrended firm size, $(K/N)_d$, decreases monotonically.

Region D Now, we turn to the case of an economy with a relatively large initial capital-output ratio, but a relatively small initial intensity of use of technological knowledge/horizontal innovation rate. The initial conditions in region D satisfy[15]

$$x(0) < x^* \wedge \quad \frac{v_{2,2}}{v_{1,2}}(x(t) - x^*) + k^* < k(0) < \frac{v_{2,1}}{v_{1,1}}(x(t) - x^*) + k^*.$$

As depicted by Figure 4, the transition is quite distinct from the one originating in region A1, with the predominance of monotonic transition paths.

Along the whole transition path, x increases commanded by the increase in the technological-knowledge stock, Q, which has an offseting effect on the horizontal entry costs, η. This happens because the initial low x determines a slow increase in the number of varieties, N, and hence a slow increase in η, whereas the large initial vertical innovation rate, I, stimulates a rapid growth of Q.

The capital-output ratio, k, which is initially high, decreases monotonically commanded by the low marginal productivity of physical capital, K, and the continuous re-allocation of resources to horizontal R&D.

In turn, the path of the technological-knowledge stock, Q, shows a very different pattern: as already said, Q rises rapidly during the early stages reflecting the high initial vertical innovation rate, I, but eventually it starts to increase at a lower rate than in the BGP as I declines monotonically (in turn reflecting the behaviour of k); as a consequence, the detrended technological-knowledge stock, Q_d, follows a hump-shaped path towards the steady state. In contrast, the detrended measure of firm size, $(K/N)_d$, shows an inverted hump-shaped behaviour. Indeed, the number of varieties, N, grows over transition at a smaller rate than in the BGP (reflecting $x(t) < x^*$) and, thus, the detrended number of varieties, N_d, falls towards the steady state. However, detrended physical capital, K_d, starts by falling by more than N_d, reflecting the fact that the initially negative g_k outweights the positive growth rate of Q; hence, $(K/N)_d$ falls and undershoots its long-run level. Then, as g_k recovers and the fall in K_d slows down, the transition-path is reversed and $(K/N)_d$ starts increasing.

Lastly, both the economic growth rate, g_Y, and the saving rate, s, follow a monotonic upward path towards the steady state driven by the monotonic increase in g_k and x.

[Figure 4 goes about here]

[15]The condition on $k(0)$ guarantees that the economy does not start in regions A2 or B1.

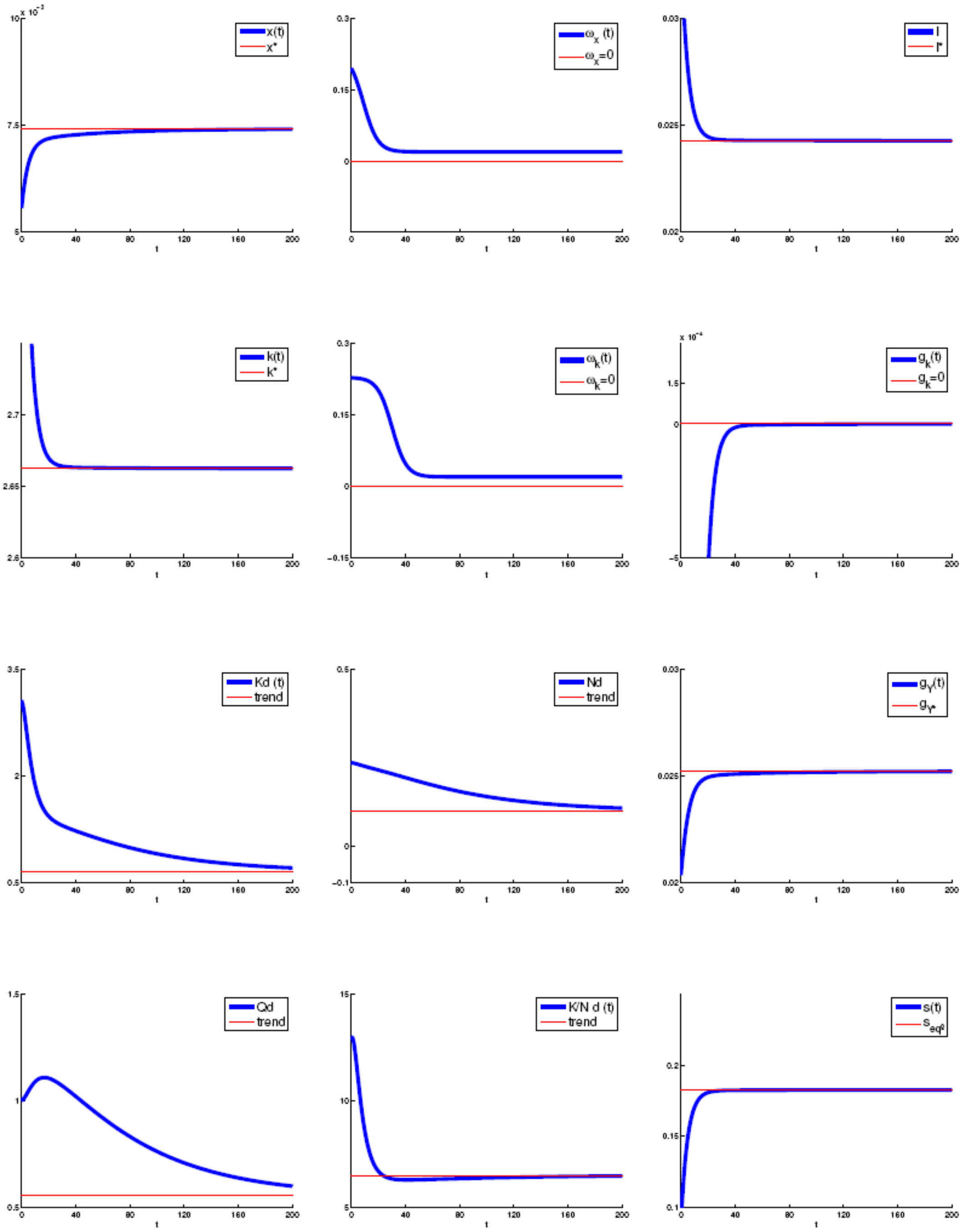

Figura 4: Transitional dynamics of the key economic variables and of the speeds of convergence of k and x (ω_k and ω_x) for an economy that sets off with $k^* < k(0)$ and $x(0) < x^*$ (region D of Figure 2). The economic growth rate, g_Y, and the saving rate, s, increase monotonically, while detrended firm size, $(K/N)_d$, displays an inverted hump-shaped behaviour.

Region B1 An economy initially in region B1 corresponds to the case of an economy with relatively low initial capital-output ratio, $k(0) < k^*$, and intensity of use of technological knowledge/horizontal innovation rate, $x(0) < x^*$. This represents the opposite scenario of an economy starting in region A1 and thus the adjustment paths, namely the non-monotonic paths of the state-like variables, are a mirror image of those observed in that region (see Figure 5).

The initial low x determines a slow increase in the number of varieties, N, and hence a slow increase in η. Nevertheless, the also low initial vertical innovation rate, I, implies an even slower increase in the technological-knowledge stock, Q, and thus x falls in the first stage of transition. The recovery of I and the consequent acceleration of Q puts x on an upward path towards the steady state.

The physical capital-output ratio, k, is initially low but rapidly increasing, induced by the high marginal productivity of physical capital, K, and the relatively low profitability of innovation activities. However, over time, the increase in the horizontal innovation rate deviates resources from the accumulation of K; eventually, the former effect is more than compensated by the latter and k starts a downfall towards the steady state. The speed of convergence, ω_k, reflects the non-monotonic transition of k, in particular the observed overshooting.

The number of varieties, N, grows over transition at a smaller rate than in the BGP (due to $x(t) < x^*$) and, thus, the detrended number of varieties, N_d, falls towards the steady state. In contrast, physical capital, K, starts by growing more than in the BGP (reflecting an initially large positive g_k) but eventually slows down to grow below the BGP; thus, detrended physical capital, K_d, observes a hump-shaped behaviour, which determines a monotonic concave upward path of detrended firm size, $(K/N)_d$.

The saving rate, s, begins by falling dramatically as the investment in horizontal R&D decreases and while vertical R&D (reflecting the vertical innovation rate, I) is still low. It experiences a slight undershoot after which it starts increasing as the allocation of resources to horizontal R&D more than compensates first the fall of the physical capital accumulation rate, g_k, and then the reduction in vertical R&D due to the falling I.

Finally, the nonlinear adjustment of the state-like variables leads once again to a nonlinear transition of the economic growth rate, g_Y. In contrast with region A1, the economic growth rate, g_Y, is initially high, because g_k is large positive, and experiences an inverted hump-shaped transition. As g_k and the horizontal innovation rate, x, decrease and the vertical innovation rate, I, is still low (although increasing), g_Y decreases rapidly and undershoots. However, the ensuing recovery of x and g_k outweights the decrease in I and allows for a gradual upward movement of g_Y towards the long-run equilibrium.

[Figure 5 goes about here]

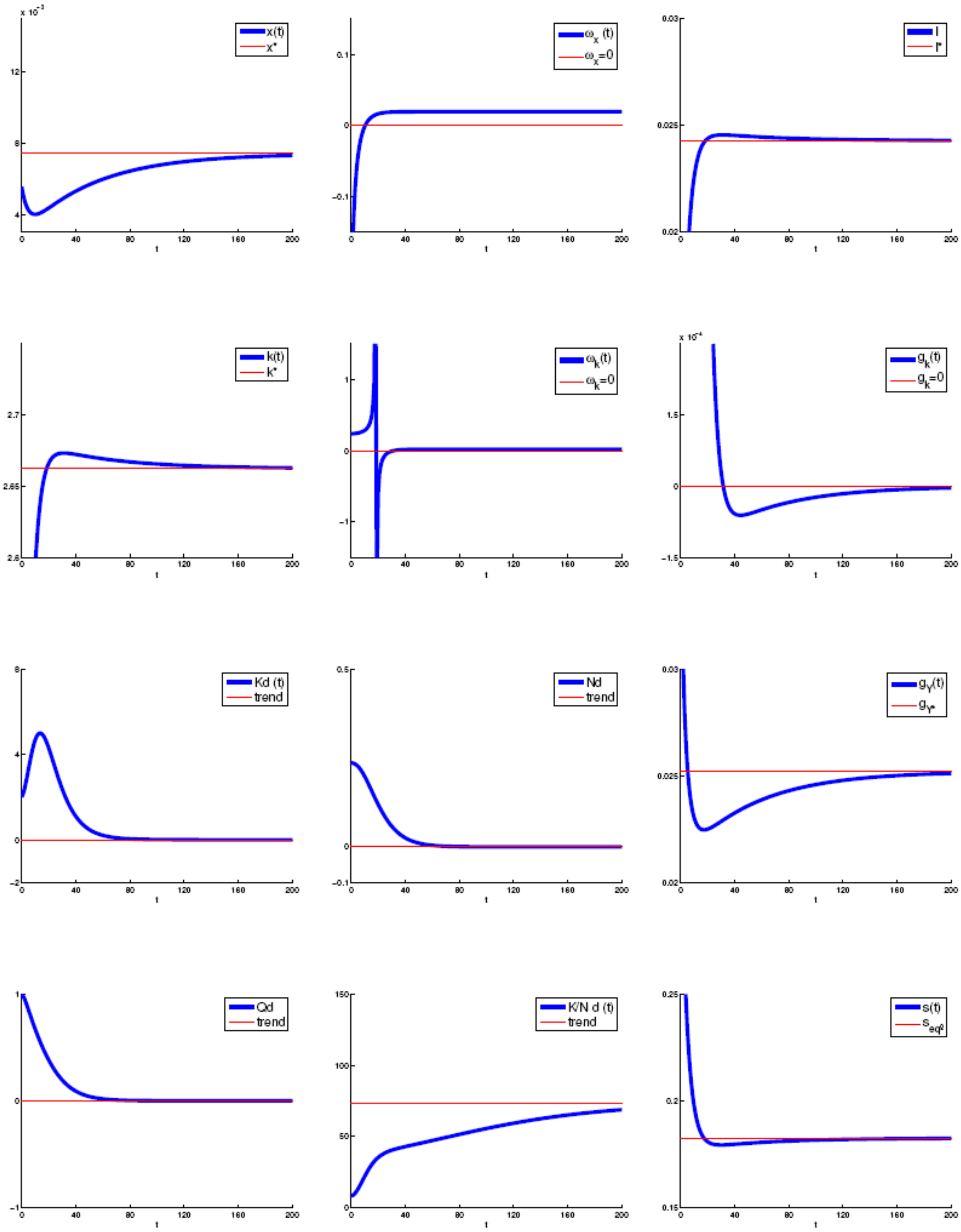

Figura 5: Transitional dynamics of the key economic variables and of the speeds of convergence of k and x (ω_k and ω_x) for an economy that sets off with $k(0) < k^*$ and $x(0) < x^*$ (region B1 of Figure 2). The economic growth rate, g_Y, and the saving rate, s, display an inverted hump-shaped behaviour, while detrended firm size, $(K/N)_d$, increases monotonically.

Region C Finally, the scenario of an economy initially with relatively low physical capital-output ratio, but a relatively large intensity of use of technological knowledge/horizontal innovation rate corresponds to an economy in region C. Initial conditions in this region satisfy[16]

$$x^* < x(0) \;\wedge\; \frac{v_{2,1}}{v_{1,1}}(x(t) - x^*) + k^* < k(0) < \frac{v_{2,2}}{v_{1,2}}(x(t) - x^*) + k^*.$$

Most aspects of the transition are a mirror image of those pertaining to region D (see Figure 6).

The monotonic decline of x towards the steady state reflects the effect of the horizontal entry costs along the whole transition path. The initial high x determines a fast increase in the number of varieties, N, and hence a fast increase in η, whereas the low initial vertical innovation rate, I, implies a slow growth of Q.

The capital-output ratio, k, which is initially low, increases monotonically commanded by the high marginal productivity of physical capital, K, and the continuous re-allocation of resources from horizontal R&D to physical investment.

The technological-knowledge stock, Q, first diverges from the BGP due to the initially low vertical inovation rate, I, but the ensuing increase in I boosts Q such that the transition is reversed; thus, in detrended terms, this variable follows an inverted hump-shaped path. In contrast, detrended firm size, $(K/N)_d$, shows a hump-shaped behaviour. Initially K/N grows above the BGP rate because the large positive g_k more than compensates for the low growth of Q and the high x (which propels N). As the increase of g_k is dampened and the innovation rate on both margins decrease (x and I), the growth rate of K/N declines towards the BGP level.

At last, both the economic growth rate, g_Y, and the saving rate, s, follow a monotonic downward path driven by the monotonic decrease in g_k and x.

[Figure 6 goes about here]

[16]The condition on $k(0)$ guarantees that the economy does not start in regions B2 or A1.

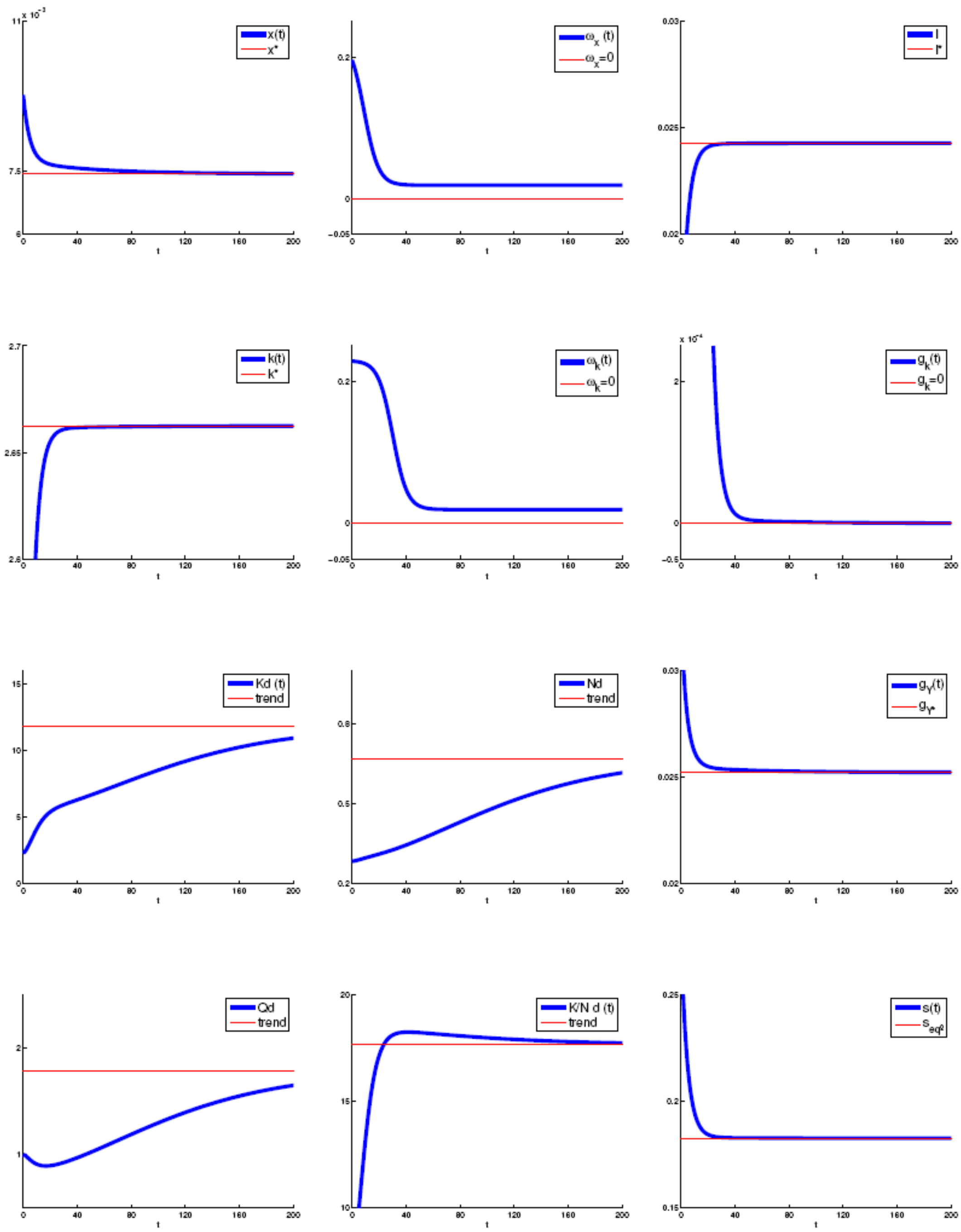

Figura 6: Transitional dynamics of the key economic variables and of the speeds of convergence of k and x (ω_k and ω_x) for an economy that sets off with $k(0) < k^*$ and $x^* < x(0)$ (region C of Figure 2). The economic growth rate, g_Y, and the saving rate, s, decrease monotonically, while detrended firm size, $(K/N)_d$, displays a hump-shaped behaviour.

5. Discussion and concluding remarks

Historical evidence of the stages of development of the nowadays most developed countries seems to suggest monotonic transitional dynamics (e.g., Maddison, 2001). However, if one looks at the development experience of a number of countries in the postwar period, the data suggests non-monotonic behaviour, through hump-shaped or inverted hump-shaped transitions (e.g., Fiaschi and Lavezzi, 2003a, 2007). Our paper develops an endogenous growth model of physical capital accumulation and two types of lab-equipment R&D featuring an analytical mechanism that encompasses all these cases.

Under this setup, both the intensive and extensive margins of growth are fully endogenous, in contrast with the existent models of the knowledge-driven type. We studied analytically the long-run equilibrium stability and transitional dynamics properties of the model, and established sufficient conditions for saddle-path stability. These conditions impose constraints only on two structural parameters, whereas the continuity property of the eigenvalues ensures that those conditions remain valid in an open set containing the specified values for these parameters.

As an alternative to the hypothesis of multiple equilibria and 'poverty traps' (e.g., Ciccone and Matsuyama, 1996; Fiaschi and Lavezzi, 2003a, 2007), our model allows us to address the above empirical evidence by emphasising the role played by an economy's initial conditions in defining the shape of its transition path towards the (unique) long-run equilibrium. We find that distinct initial physical capital-output ratios and/or initial technology intensities in otherwise similar economies may imply contrasting – either monotonic or non-monotonic – convergence paths of key economic variables, namely the economic growth rate, saving rate, and firm size (assets per firm), as emphasised by the empirical literature.

In particular regarding the transition of growth rates, the evidence reported by Maddison (2001) (monotonic downward transition) corresponds, in our model, to the case of the economies with initial below-the-frontier per capita output and that start in region C of the phase diagram, characterised by an initial relative scarcity of physical inputs – i.e., of physical capital, but also of varieties of technological goods (implying a relatively high technology intensity). Therefore, our model specialises the explanation proposed by the neoclassical growth model for this sort of transition, only focused on physical capital. The evidence found by Fiaschi and Lavezzi (2003a, 2007) for initially 'low' and 'middle-low' income countries (respectively, inverted hump-shaped and hump-shaped transitions) corresponds, in the model, to the case of the economies also with initial below-the-frontier per capita output but that start, respectively, in regions B1 and A1 of the phase diagram. In the former case, the economies are characterised by an initial relative scarcity of physical capital but a relative abundance of varieties of technological goods (implying a relatively low technology intensity). These initial conditions determine a first phase of high growth (take-off), as in the neoclassical growth model, but which is followed by a phase of quite lower growth rates (i.e., an undershooting phase, which, if long-lasting, may be perceived as a 'poverty trap') and, eventually, by a slow recovery into a regime of moderate growth. In the other case (region A1), the opposite occurs: initial conditions favour the exploration of the complementarity between physical capital and technology intensity, and thus a transition occurs in which growth rates do not peak at the beginning of the convergence process but later on; hence, the overshooting of the growth rate, later followed by a slowdown towards a regime of moderate growth.

The model also predicts speeds of convergence that vary over time and across variables, in particular with physical capital and knowledge displaying contrasting speeds of convergence, in line with the empirical evidence emphasised by, e.g., Bernard and Jones (1996).

Our theoretical model allows for technological-knowledge and physical capital accumulation, but no human capital accumulation. However, since, in our model, vertical and horizontal R&D are complementary, a possible extension is to replace vertical R&D with human capital accumulation, in order to explore the complementarity between human capital and technology accumulation, in line with Nelson and Phelps (1966), Arnold (1998), Kosempel (2004), and others. Then, our equation (18) implies that the growth rate of the number of varieties (i.e., the rate of technology accumulation in this version of the model) is a positive function of the stock of human capital, whereas reinterpreting equation (19) as a human-capital accumulation function implies that the accumulation of human capital is a negative function of the existing number of varieties (i.e., the technology stock in this version of the model). The idea here is that a higher technology stock makes learning more complex and demanding, in line with, e.g., Galor and Moav (2002) and Reis and Sequeira (2007). This modification to our model shows that technology intensity is isomorphic to the human capital-technology stock ratio and, thus, a country's initial conditions can alternatively be interpreted in terms of the physical capital-output and the human capital-technology stock ratios.

Overall, these results suggest that institutions may be relevant for a country's growth experience also by playing a role in the determination of the initial endowment of physical versus immaterial inputs: on one hand, physical capital versus technological-knowledge stock; on the other, the proportion of the extensive versus the intensive (or human-capital) component of a given technological-knowledge stock. In this context, not only may institutions be capable of influencing the timing of the economies' take-off (see, e.g., Jones and Romer, 2010), but also the characteristics of their transition (speed and shape) towards the respective long-run equilibrium.

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Appendix

A. General equilibrium: derivation of equation (24)

Consider the households' balance sheet

$$a(t) = K(t) + \eta(t) \cdot N(t) \tag{39}$$

Hence, we can characterise the change in the value of equity as

$$\dot{a}(t) = \dot{K}(t) + \dot{\eta}(t) \cdot N(t) + \eta(t) \cdot \dot{N}(t) \tag{40}$$

Substitute (21) in the left-hand side of (40) and $\frac{\dot{\eta}}{\eta} = \frac{\dot{Q}}{Q} - \frac{\dot{N}}{N} = I \cdot \left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)$ – derived from (16) and (19) – in the right-hand side, to get

$$(r(t)+I(t))\cdot\eta(t)\cdot N(t)-I(t)\cdot\eta(t)\cdot N(t)+r(t)\cdot K(t(t))+w(t)\cdot L-C(t)=$$

$$=\dot{K}(t)+I(t)\cdot\left(\lambda^{\frac{\alpha}{1-\alpha}}-1\right)\cdot\eta(t)\cdot N(t)+\eta(t)\cdot\dot{N}(t) \tag{41}$$

Since the real interest rate r consists of dividend payments in units of asset price minus the Poisson death rate, i.e., $r=\frac{\pi}{V}-I$, for each t, then $\pi\cdot N=(r+I)\cdot V\cdot N$. Furthermore, it is easy to show, bearing in mind (1), (4) and (6), that $w\cdot L=(1-\alpha)\cdot Y$, $\Pi=\pi\cdot N=(\alpha-\alpha^2)\cdot Y$ and $r\cdot K=\alpha^2\cdot Y$. Using these results in (41), yields

$$(\alpha-\alpha^2)\cdot Y(t)+\alpha^2\cdot Y(t)+(1-\alpha)\cdot Y(t)-C(t)=$$

$$=\dot{K}(t)+\left[I(t)+I(t)\cdot\left(\lambda^{\frac{\alpha}{1-\alpha}}-1\right)\right]\cdot\eta(t)\cdot N(t)+\eta(t)\cdot\dot{N}(t) \tag{42}$$

Next, recall that $R_n=\eta\dot{N}$ and $R_v=\lambda^{\frac{\alpha}{1-\alpha}}\cdot\zeta\cdot L\cdot I\cdot Q$, such that (42) reads as (24) in the text.